Deliverable 3.1: Identifiability of semi-blind methods

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Abstract

Deliverable 3.1 summarizes the activities and scientific results of Task 3.1, which focuses on the analysis of identifiability conditions for semi-blind channel estimation and data detection in centralized Cell-Free (CF) Massive MIMO systems with joint processing at a central unit. Building on the deterministic identifiability results originally established in [1], the CF Massive MIMO system is represented through an associated bipartite graph, enabling the formulation of a message-passing algorithm that jointly verifies identifiability and allows for the exact recovery of both user channels and transmitted data when identifiability holds.

Leveraging the observation that semi-blind identifiability is equivalent to the absence of a Karp–Sipser core in the associated bipartite graph, we study identifiability for an ensemble of CF networks generated by spatial Poisson Point Processes (PPPs) in the asymptotic regime where the network dimension tends to infinity. By relaxing the geometric edge dependencies inherent to PPP-based bipartite graphs, we derive a tractable density evolution recursion that depends only on macroscopic network parameters, namely the user and access point spatial intensities λ_T and λ_R, and the neighborhood radius γ.

The resulting density evolution analysis reveals a sharp phase transition in the (λ_T, λ_R) parameter space, separating PPP-based network ensembles that are identifiable with high probability from those that are not. These results provide a macroscopic characterization of identifiability for semi-blind methods in large-scale CF Massive MIMO systems and offer valuable insights for system design and dimensioning.

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Contents

1 Introduction

Cell-Free (CF) Massive MIMO has emerged as a promising architecture for future wireless systems, particularly in the context of beyond-5G and 6G networks. By deploying a very large number of geographically distributed access points (APs) connected to a central processing unit (CPU), CF Massive MIMO systems enable joint processing of uplink and downlink signals over large areas, offering improved coverage, macro-diversity, and spectral efficiency compared to conventional cellular deployments.

Despite these advantages, the performance of CF Massive MIMO systems is severely limited by pilot contamination, which arises from the reuse of pilot sequences during channel estimation. Unlike centralized massive MIMO systems, where favorable propagation and channel hardening can be exploited to mitigate pilot contamination, these properties are significantly weakened in distributed CF architectures. As a consequence, pilot contamination remains a fundamental bottleneck and has motivated extensive research on pilot assignment, pilot reuse, and interference-aware processing techniques, see e.g., the related discussion on the state of the art in [2].

An alternative and complementary approach to pilot-based methods is provided by blind and semi-blind channel estimation and data detection techniques, which exploit the inherent structure and sparsity of the channel coefficients in CF Massive MIMO systems. In this context, the notion of identifiability plays a central role. Identifiability characterizes whether the unknown channel coefficients and transmitted data symbols can be uniquely recovered from the received signals in the absence of noise, and it guarantees the non-singularity of the Fisher Information Matrix and the existence of the Cramér–Rao bound. As such, identifiability conditions provide fundamental insights into the feasibility limits of semi-blind processing.

In a previous study [1], we derived deterministic sufficient and necessary conditions for semi-blind identifiability in CF Massive MIMO systems by explicitly exploiting the sparsity of the channel support induced by path loss and limited propagation range. These conditions admit a natural graph-theoretic interpretation, in which the system is represented by a bipartite graph connecting users and APs through their local neighborhoods. In that work, identifiability was characterized in terms of the existence of specific structural properties of this graph.

The present deliverable leverages these earlier results and extends them in several important directions. First, although the identifiability conditions were already established in [1], they are recalled and presented in detail in Section 3 for the convenience of the reader and to provide a self-contained exposition. This detailed presentation is instrumental for the subsequent developments, as it enables a transparent and systematic derivation of an explicit message-passing algorithm. This algorithm not only verifies whether the identifiability conditions are satisfied, but also provides a constructive procedure to recover exactly the channel coefficients and transmitted data whenever identifiability holds.

Second, leveraging the observation that semi-blind identifiability is equivalent to the absence of a residual Karp–Sipser core in the associated bipartite graph, we investigate identifiability in a statistical network model based on spatial Poisson Point Processes (PPPs). In this framework, users and APs are distributed according to independent PPPs, and edges in the bipartite graph are induced by a finite neighborhood radius. This modeling choice allows us to study identifiability as a function of macroscopic network parameters, such as user and AP spatial intensities λ_T and λ_R, respectively, and the neighborhood radius γ, independently of any particular network realization.

Finally, by relaxing the geometric dependencies among edges in the PPP-based bipartite graph, we introduce an analytically tractable surrogate ensemble with independent edges but identical local degree statistics. This approximation enables the derivation of a density evolution recursion, analogous to that used in the analysis of low-density parity-check (LDPC) codes over the binary erasure channel. The resulting analysis reveals a sharp phase transition between identifiable and non-identifiable regimes in the large-system limit and leads to a precise characterization of the identifiability region in the system parameter space.

The remainder of this deliverable is organized as follows. Section 2 introduces the CF Massive MIMO system model. Section 3 reviews the deterministic identifiability conditions previously derived in [1], which are included here for completeness and to support the development of the

message-passing algorithm. Section 4 presents the message-passing procedure for identifiability verification and exact parameter recovery. Section 5 introduces the PPP-based network model and the associated bipartite graph ensemble. Section 6 analyzes asymptotic identifiability for PPP networks via density evolution. Section 7 characterizes the identifiability region and highlights the associated phase-transition behavior.

Notation: In the following, superscripts ^{T}, *, and ^{H} stand for transpose, conjugate and conjugate transpose, respectively. Uppercase and lowercase bold symbols are used to denote matrices and vectors, respectively. The expectation operator is indicated by $\mathbb{E}\{.\}$ and $\mathbf{I}_P$ is the $P \times P$ identity matrix. Here, $\|\cdot\|$ and diag(.) denote the Euclidean norm operator and the squared diagonal matrix consisting of the diagonal elements of matrix argument, respectively. The operator $\Re\{.\}$ yields the real part of its argument whereas vec(.) denotes $\text{vec}(\mathbf{A}) = \left[\mathbf{A}_{:,1}^T \ \mathbf{A}_{:,2}^T \cdots \mathbf{A}_{:,n}^T\right]^T$, where $\mathbf{A}_{:,j}$ is the j-th column of matrix $\mathbf{A}$. Finally, $\mathcal{N}(\mu, \sigma^2)$ and $\mathcal{CN}(\mu, \sigma^2)$ denote a real and a complex Gaussian distribution with mean μ and variance σ^2, respectively.

2 System Model

We consider the uplink of a CF MaMIMO system consisting of K users and M APs equipped with a single antenna and randomly distributed over a $D \times D$ square area. We assume that $M \geq K$. The M APs are connected to a central processing unit (CPU) via a back-haul network. The channel matrix between the APs and users is given by $\mathbf{H} \in \mathbb{C}^{M \times K}$, whose (m,k)-element h_{mk} is the channel coefficient between AP m and user k and is modeled as follows

$$h_{mk} = \sqrt{\beta_{mk}}\, g_{mk}, \tag{1}$$

where β_{mk} represents the large-scale fading coefficient which accounts for path loss and shadowing effects and g_{mk} represents the small-scale fading. We assume that g_{mk}, $m = 1, \cdots M$, $k = 1, \cdots K$, are independent and identically distributed (i.i.d.) complex normal random variables, i.e., $g_{mk} \sim \mathcal{CN}(0,1)$. Additionally, we assume perfect knowledge of the large-scale fading coefficients β_{mk}, $m = 1, \cdots M$, $k = 1, \cdots K$ at the CPU. Due to the path loss, the channel coefficients are assumed to be negligible at distances higher than a given threshold γ. Then, for each AP m, the CPU is required to estimate only the channels of the users in a disc centered around AP m with radius γ while the signals transmitted from users external to the disc are treated as additive noise. We denote by $\mathcal{K}_I(m)$ and $\mathcal{K}_0(m)$ the sets of users inside the disc centered around AP m and remaining users, respectively. At a global level, this determines a partition of the channel coefficients into two groups, the channel coefficients that have to be detected $\mathcal{K}_I \equiv \{(m,k) | m = 1, \dots M, k \in \mathcal{K}_I(m)\}$ and the complement set $\mathcal{K}_0 \equiv \{(m,k) | m = 1, \dots M, k \in \mathcal{K}_0(m)\}$. Consistently with this partition, we decompose the channel matrix $\mathbf{H}$ into two matrices $\mathbf{H}_I$ and $\mathbf{H}_0$ such that $\mathbf{H} = \mathbf{H}_I + \mathbf{H}_0$. Then, $\mathbf{H}_I$ and $\mathbf{H}_0$ of size $M \times K$ denote the matrices of the relevant and negligible channel coefficients, respectively. Throughout this paper, we assume that $\gamma \ll D$ and the APs are distributed over the whole region such that matrix $\mathbf{H}_I$ has a large number of zero elements.

In the uplink transmission, each user sends one of P pilot sequences known by the CPU followed by $L - P$ unknown data symbols. The pilot sequences are assumed to be ortho-normal, i.e., orthogonal with unit norm. The L received symbols at the M APs are given by

$$\mathbf{Y} = \sqrt{\rho}\, \mathbf{H}_I\, \mathbf{X} + \sqrt{\rho}\, \mathbf{H}_0\, \mathbf{X} + \mathbf{W}, \tag{2}$$

where ρ denotes the transmit power at each user terminal normalized by the noise variance. $\mathbf{Y} \in \mathbb{C}^{M \times L}$ is a matrix of the L received symbols at the M APs and $\mathbf{X} \in \mathbb{C}^{K \times L}$ is a matrix of the transmitted symbols. Note that the k-th row corresponds to the signals transmitted by user k. The matrix $\mathbf{W} \in \mathbb{C}^{M \times L}$ is the additive white Gaussian noise (AWGN) with i.i.d. components having zero mean and unit variance.

Let $\mathbf{X}_p \in \mathbb{C}^{K \times P}$ and $\mathbf{X}_d \in \mathbb{C}^{K \times (L-P)}$ denote the pilot sequences and data symbols, respectively. Then, $\mathbf{X} = [\mathbf{X}_p \ \ \mathbf{X}_d]$. Similarly, $\mathbf{Y} = [\mathbf{Y}_p \ \ \mathbf{Y}_d]$ where $\mathbf{Y}_p \in \mathbb{C}^{M \times P}$ and $\mathbf{Y}_d \in \mathbb{C}^{M \times (L-P)}$ represent

the matrices of received training and data signals, respectively.

3 Identifiability

In this section, we derive sets of both sufficient and necessary conditions for the identifiability of vector parameter $\boldsymbol{\theta}$ under the assumption that $\boldsymbol{\theta}$ is a deterministic unknown parameter. Then, we propose an MP algorithm over a graph that determines the exact channel coefficients if the sufficient identifiability conditions are satisfied. Finally, we show that the system is identifiable via semi-blind algorithms if the Karp-Sipser algorithm applied to the same graph yields an empty core paving the way to an analysis of asymptotically large networks based on core percolation properties.

In the framework of deterministic identifiability, we assume that vector parameter $\boldsymbol{\theta}$ is deterministic and consider channel $\mathbf{H}_0$ negligible. Then, the observation $\mathbf{y}$ is Gaussian distributed, i.e., $\mathbf{y} \sim \mathcal{CN}(m_{\mathbf{y}}(\boldsymbol{\theta}), \mathbf{I}_{ML})$ with covariance matrix independent of $\boldsymbol{\theta}$. The identifiability of $\boldsymbol{\theta}$ relies only on the known mean $m_{\mathbf{y}}(\boldsymbol{\theta})$ and, for semi-blind methods, $\mathbf{X}_d$ and $\mathbf{h}_I$ are said to be *identifiable* [3] if

$$\mathbf{H}_I\mathbf{X} = \mathbf{H}_I^{'}\mathbf{X}^{'} \;\Rightarrow\; \mathbf{h}_I = \mathbf{h}_I^{'} \text{ and } \mathbf{X}_d = \mathbf{X}_d^{'} \tag{3}$$

Let $m_{\mathbf{Y}^{\sim 0}}$ be the expectation of $\mathbf{Y}$ in (2) obtained assuming $\mathbf{H}_0$ negligible. The identifiability problem reduces to analyze the following bi-linear system of equations in the unknowns $\mathbf{h}_I$ and $\mathbf{X}_d$

$$m_{\mathbf{Y}^{\sim 0}} = \sqrt{\rho}\,\mathbf{H}_I\mathbf{X}$$

and determine under which conditions this system admits a unique solution, which is assumed to exist. These identifiability conditions are summarized in the following proposition.

Proposition 1. ***Sufficient Identifiability Conditions*** *– Let $\mathcal{S}_k$ denote the support of the channel of user k, i.e., the set of all the indices m such that $\mathbf{H}_{I,m,k} \neq 0$, and let $|\mathcal{S}_k|$ be its cardinality. In a semi-blind joint data detection and channel estimation method, the unknown parameters $\mathbf{h}_I$ and $\mathbf{X}_d$ are identifiable if (i) the $K \times L$ matrix $\mathbf{X}$, with $L \geq K$ has full row rank K, (ii) the channel of each user is sparse and $|\mathcal{S}_k| \leq M - K + 1$, and (iii) for each group of users $\mathcal{G}_p$ utilizing the same ortho-normal pilot sequence $\mathbf{x}_p^{(p)}$, it is possible to identify a sequence $\{\mathcal{G}_{p,1}, \mathcal{G}_{p,2}, \ldots \mathcal{G}_{p,s}\}$ satisfying the following properties:*

1. *$\bigcup_{j=1}^{s} \mathcal{G}_{p,j} \equiv \mathcal{G}_p$, i.e., the sequence of subsets is a partition of $\mathcal{G}_p$*
2. *In the support of the channel of each user $k \in \mathcal{G}_{p,i}$ there exists at least an index $j \in \mathcal{S}_k$ that is not contained in any of the channel supports of other users in the same group $\mathcal{G}_{p,i}$ or in the following groups of the sequence $\mathcal{G}_{p,i+1}, \ldots \mathcal{G}_{p,s}$.*

Remark 1. *Condition (iii)-2 implies that the signal transmitted by each user k in $\mathcal{G}_{p,i}$ impinges an AP in the disc $\mathcal{M}_k$ centered around user k with radius γ and no other signal transmitted by other users in $\mathcal{G}_{p,i}$ or subsequent subsets $\mathcal{G}_{p,i+1}, \mathcal{G}_{p,i+2}, \ldots \mathcal{G}_{p,s}$ impinges the same AP.*

Remark 2. *The assumption that $\mathbf{X}$ has full row rank K implies that $\mathbf{X}_d$ has at least rank $K - P$.*

Proof: Observe that since in CF MaMIMO systems $M \gg K$, and the channel matrix $\mathbf{H}_I$ consists of independent channels, we can assume that it has full row rank equal to K with probability 1. Thanks to the assumptions of Proposition 1, also matrix $\mathbf{X}$ has full row rank equal to K as well as matrix $m_{\mathbf{Y}^{\sim 0}}$. Then, the singular value decomposition (SVD) of the noise-free system is given by

$$\frac{1}{\sqrt{\rho}}\, m_{\mathbf{Y}^{\sim 0}} = \mathbf{H}_I\mathbf{X} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^H \tag{4}$$

where $\mathbf{U} \in \mathbb{C}^{M \times K}$ and $\mathbf{V} \in \mathbb{C}^{L \times K}$ are the matrices of the left and right singular-vectors and $\boldsymbol{\Sigma}$ is the $K \times K$ diagonal matrix of singular values. Additionally, the left and right singular value

matrices $\mathbf{U}$ and $\mathbf{V}$ span the channel subspace $\mathbf{H}_I$ and the signal space $\mathbf{X}$, respectively. Then, the problem of identifiability reduces to determine a $K \times K$ non-singular matrix $\mathbf{T}$ such that $\mathbf{H}_I = \mathbf{U}\mathbf{T}$ and then, also matrix $\mathbf{X}$ is unequivocally given by $\mathbf{X} = \mathbf{T}^{-1}\mathbf{\Sigma}\mathbf{V}^H$. In order to determine matrix $\mathbf{T}$, we utilize the following properties and information:

- The support of each user channel is known and sparse and at least $K-1$ channel coefficients are zero.
- The *contaminated channel.* More specifically, let us consider the linear system of equations corresponding to the transmission of the pilot sequences, i.e., $\frac{m_{\mathbf{Y}_p^{\sim 0}}}{\sqrt{\rho}} = \mathbf{H}_I \mathbf{X}_p$, where $m_{\mathbf{Y}_p^{\sim 0}}$ denotes the expectation of $\mathbf{Y}_p = \sqrt{\rho}\mathbf{H}_I\mathbf{X}_p$. By post-multiplying both sides of the system by the pilot sequence $\mathbf{x}_p^{(p)}$ and exploiting the ortho-normality of the training sequences, $\mathbf{X}_p\, \mathbf{x}_p^{(p)} = \mathbf{1}_{\mathcal{G}_p}$ where $\mathbf{1}_{\mathcal{G}_p}$ is the K-dimensional vector with elements with indices in $\mathcal{G}_p$, i.e., indices corresponding to users transmitting pilot $\mathbf{x}_p^{(p)}$, equal to one and and zero elsewhere. Then, it is apparent that this system of equations enables to determine exactly at each AP the sum of all the non-zero channel coefficients of the users in each group $\mathcal{G}_p$, $p = 1, \dots P$, i.e., $\frac{1}{\sqrt{\rho}}\, m_{\mathbf{Y}_p^{\sim 0}}\, \mathbf{x}_p^{(p)} = \mathbf{H}_I\, \mathbf{1}_{\mathcal{G}_p}$.

Then, let us focus on a user k in $\mathcal{G}_{p,1}$. Thanks to the assumptions on the partition of $\mathcal{G}_p$, there exists at least an AP m such that $\mathbf{H}_{I,m,:}\, \mathbf{1}_{\mathcal{G}_p} = h_{m,k} = \frac{1}{\sqrt{\rho}}\, m_{\mathbf{Y}_p^{\sim 0}}\, \mathbf{x}_p^{(p)} \neq 0$, where $\mathbf{H}_{I,m,:}$ denotes the m-th row of the matrix $\mathbf{H}_I$. Furthermore, thanks to the assumption on the sparsity of the channels, we can obtain from the system of equations $\mathbf{H}_{I,:,k} = \mathbf{U}\mathbf{T}_{:,k}$ $K-1$ equations where the channel of user k is zero. Then, we can construct a non-homogeneous system of equations in the unknown $\mathbf{T}_{:,k}$ and the vector of constant terms consisting of zeros and at least the non-zero element h_{mk}. This system can be unequivocally solved to determine $\mathbf{T}_{:,k}$. Thanks to the properties of the sequence $\mathcal{G}_{p,1}, \mathcal{G}_{p,2}, \dots \mathcal{G}_{p,s}$, it is possible to determine sequentially, the columns of matrix $\mathbf{T}$ corresponding to a certain group, compute exactly the corresponding channels of the users in the group and cancel them from the contaminated channel for group $\mathcal{G}_p$ until the complete computation of all the columns of matrix $\mathbf{T}$ corresponding to all the users in $\mathcal{G}_p$ and the corresponding channels. This approach can be repeated for all the groups up to the complete computation of matrix $\mathbf{T}$ and channel $\mathbf{H}_I$. Then, we observe that $\mathbf{T}$ has full rank K since $\mathbf{H}_I$ has full row rank. The inverse of $\mathbf{T}$ exists and enables the computation of $\mathbf{X}_d$. This concludes the proof. ■

In the following, let $(\mathbf{H})_{\mathcal{G}_p}$ denote a reduced version of the matrix $\mathbf{H}$ containing only the columns corresponding to the users in $\mathcal{G}_p$.

Proposition 2. ***Necessary Identifiability Conditions*** *– Identification of* $\mathbf{h}_I$ *and* $\mathbf{X}_d$ *from the product* $\mathbf{H}_I\, \mathbf{X}$ *leads to the global necessary identifiability condition*

$$\frac{1}{K} \sum_{k=1}^{K} |\mathcal{S}_k| \;\leq\; M - K + P \tag{5}$$

or the per pilot necessary identifiability condition

$$\frac{1}{|\mathcal{G}_p|} \sum_{k \in \mathcal{G}_p} |\mathcal{S}_k| \;\leq\; M - K + \frac{K}{|\mathcal{G}_p|} \quad p = 1, \dots P\,. \tag{6}$$

Proof: Consider again the SVD in (4), $\mathbf{H}_I\mathbf{X} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^H$, with $\mathbf{V}^H$ partitioned into P plus $L-P$ columns similar to $\mathbf{X}$, $\mathbf{V}^H = [\mathbf{V}_p^H\ \mathbf{V}_d^H]$. Introducing again the unknown $K \times K$ mixture $\mathbf{T}$, this leads to the equations

$$\mathbf{H}_I = \mathbf{U}\,\mathbf{T}\,, \quad \mathbf{T}\,\mathbf{X}_p = \mathbf{\Sigma}\,\mathbf{V}_p^H \tag{7}$$

which together represent $K(M+P)$ equations in the $\sum_{k=1}^{K} |\mathcal{S}_k|$ unknowns $\mathbf{h}_I$ and the K^2 unknowns $\mathbf{T}$. The proper conditions for solvability of the equations (7), that the number of equations needs

to be at least equal to the number of unknowns, then leads to (5). If now we consider the equations for group of users $\mathcal{G}_p$, multiplying $\mathbf{T}\,\mathbf{X}_p = \mathbf{\Sigma}\,\mathbf{V}_p^H$ by $\mathbf{x}_p^{(p)}$ and exploiting $\mathbf{X}_p\,\mathbf{x}_p^{(p)} = \mathbf{1}_{\mathcal{G}_p}$ then we get

$$(\mathbf{H}_I)_{\mathcal{G}_p} = \mathbf{U}\,(\mathbf{T})_{\mathcal{G}_p}\ ,\quad \mathbf{T}\,\mathbf{1}_{\mathcal{G}_p} = (\mathbf{T})_{\mathcal{G}_p}\,\mathbf{1} = \mathbf{\Sigma}\,\mathbf{V}_p^H\mathbf{x}_p^{(p)} \tag{8}$$

which represents $M\,|\mathcal{G}_p| + K$ equations in the $\sum_{k\in\mathcal{G}_p}|\mathcal{S}_k| + K\,|\mathcal{G}_p|$ unknowns in $(\mathbf{H}_I)_{\mathcal{G}_p}$ and $(\mathbf{T})_{\mathcal{G}_p}$, thus leading to (6). ■

It should be noted that verifying the sufficient conditions for identifiability of the deterministic parameters stated in Proposition 1 is nontrivial. In particular, identifying a sequence of disjoint subsets $\{\mathcal{G}_{p,1}, \mathcal{G}_{p,2}, \ldots \mathcal{G}_{p,s}\}$ that satisfies Conditions 1 and 2 of the proposition requires a systematic and well-defined procedure.

The proof of Proposition 1 also provides a constructive approach for determining such sequences and for recovering the unknown parameters $\mathbf{H}_I$ and $\mathbf{X}_d$ provided that, for each $p = 1, \ldots P$, there exists a suitable sequence $\{\mathcal{G}_{p,1}, \mathcal{G}_{p,2}, \ldots \mathcal{G}_{p,s}\}$ that forms a partition of the set $\mathcal{G}_p$.

In the following section, we present an iterative algorithm that determines whether such partition sets exist and, at the same time, enables the computation of the unknown parameters.

4 Message Passing for Identifiability Check and Parameter Computation

In this section, building on the proof of Proposition 1 we provide a Message Passing algorithm that enables to identify at iteration i the set $\mathcal{G}_{p,i}$ if the set exists, and determine the channel coefficients of all users in the set.

To address this problem, we focus on the set $\mathcal{G}_p$ of users transmitting pilot sequence p.

We construct a graph by associating a variable node with each user k and a factor node with each AP m. A bipartite graph is then formed by connecting a variable node to a factor node whenever the distance between the corresponding user and AP is below a threshold γ.

We further assume that the factor nodes are initialized with the values of the vector $\mathbf{h}_{I,p}^c = \mathbf{H}_I\mathbf{1}_{\mathcal{G}_p}$, that is, the sum of the channel coefficients of all users within the corresponding γ-neighborhood. Each variable node knows the matrix $\mathbf{U}$ that spans the channel subspace. Throughout this section, we denote by $\mu_{k\to m}^p$ the message transmitted by variable node k to factor note m and by $\nu_{m\to k}^p$ the message transmitted by factor node m to variable node k.

The initial step of the message passing algorithm starts at the factor nodes. Each factor node m that is a leaf transmits its initialization value $h_{I,p,m}^c$ to its neighbor. It transmits an erasure Δ if it is not a leaf, that is,

$$\nu_{m\to k}^p = \begin{cases} h_{I,p,m}^c & \text{Factor node } m \text{ is a leaf and } k \text{ is its neighbor} \\ \Delta & \text{Factor node } m \text{ is not a leaf} \end{cases} \tag{9}$$

.

At iteration i, each variable node k that has received at least a message that is not an erasure solves the system of equations $\mathbf{U}\mathbf{T}_{:,k} = \mathbf{H}_{I,:,k}$ utilizing that value. The construction of a system of K equations to determine $\mathbf{T}_{:,k}$ is detailed in the proof of Proposition 1 and exploits the channel sparsity. Once $\mathbf{T}_{:,k}$ is known, it is possible to determine all the non-zero channel coefficients $\mathbf{H}_{I,:,k}$. Then, variable node k transmits to all its neighbors the corresponding channel coefficients. Variable node k transmits the same messages in all the following iterations. If variable node k receives all erasures it transmits erasures to all its neighbors. The message is then defined as follows:

$$\mu_{k\to m}^p = \begin{cases} \mathbf{H}^{I,m,k} & \exists m' \text{ such that } \nu_{m'\to k}^p \neq \Delta \\ \Delta & \forall m' \nu_{m'\to k}^p = \Delta \end{cases} \tag{10}$$

.

The second step of iteration i determines the messages at the factor nodes. A factor node m computes a message for the output edge $< m, k >$ as the difference between its initialization value

$h^c_{I,p,m}$ and all the incoming messages. The resulting message is not an erasure if all the incoming messages are not erasures otherwise the factor node transmits an erasure. Then, message $\nu^p_{m\to k}$ is given by

$$\nu^p_{m\to k} = \begin{cases} [\mathbf{h}^c_{I,p}]_m - \sum_{k'\neq k} \mu^p_{k'\to m} & \forall k' \neq k : \mu^p_{k'\to m} \neq \Delta \\ \Delta & \exists k' \neq k : \mu^p_{k'\to m} = \Delta \end{cases}, \tag{11}$$

where $[\mathbf{h}^c_{I,p}]_m$ denotes the m-th element of vector $[\mathbf{h}^c_{I,p}]_m$.

Set $\mathcal{G}_{p,i}$ includes all users/variable nodes that compute their channel coefficients at iteration i.

The Message Passing algorithm ends when all the channel coefficients have been determined and in this case the identifiability conditions are satisfied or when no additional erasure can be determined and thus the system is not identifiable.

Interestingly, this algorithm is closely related to the MP algorithm for decoding of low density parity check codes (LDPCs) in transmissions through binary erasure channels in [4]. The similarities and differences between the proposed message-passing procedure for identifiability verification and the message-passing algorithm used for LDPC decoding over binary erasure channels are discussed and analyzed in the following section.

Additionally, let us consider the Karp-Sipser or greedy leaf removal procedure [5–7] which consists in removing from a graph sequentially all the leaves and corresponding edges and observe that sequential or simultaneous removal of leaves is equivalent in asymptotic conditions. Then, the sufficient identifiability conditions in Proposition 1 are satisfied if the greedy leaf removal procedure yields an empty core.

5 Poisson Point Processes Networks

In this section, we introduce a Poisson Point Process (PPP)–based network model for the statistical characterization of a CF MaMIMO system. This framework enables the analysis of the statistical properties of the bipartite graph associated with the network. Based on this model, we define a PPP network ensemble and use it to study the identifiability properties of the ensemble.

We consider a network defined over a square region of side length D and introduce a random ensemble of PPP networks in the asymptotic regime as $D \to +\infty$. The APs and the users are modeled as points of independent homogeneous PPPs. In particular, the APs form a PPP with intensity λ_R, i.e., on average there are λ_R APs per unit area.

Users transmitting the same pilot sequence $\mathbf{x}^{(p)}_p$ are modeled as points of a homogeneous PPP with intensity $\lambda^{(p)}_T$. By the superposition property of PPPs, the union of the user point processes associated with different pilot sequences is also a PPP with intensity

$$\lambda_T = \sum_{p=1}^{P} \lambda^{(p)}_T. \tag{12}$$

We define the γ-neighborhood of a point $\mathfrak{P}$ belonging to the AP PPP as the set of all the points of the user PPP that lie within a distance lower than γ from $\mathfrak{P}$. In a dual way, we can define the γ-neighborhood of a point $\mathfrak{P}$ belonging to the user PPP.

The user and AP PPPs, together with the notion of a γ-neighborhood, induce a random bipartite graph in which the two disjoint node sets correspond to the AP PPP and the user PPP, respectively. An edge exists between an AP node and a user node if and only if the AP lies within the γ-neighborhood of the user, and equivalently, the user lies within the γ-neighborhood of the AP.

We refer to as the standard ensemble of PPP networks with parameters $(\lambda_T, \lambda_R, \gamma)$ the ensemble of random bipartite graphs or, equivalently, random PPP networks for increasing values of the area D.

By adopting the same denomination utilised in Section 4, we refer to the nodes in the user PPP as variable nodes and the nodes in the AP PPP as factor nodes.

Based on the properties of PPP processes we can state the following properties:

- For any bounded set $\mathcal{A} \in \mathbb{R}^2$, the number of points in $\mathcal{A}$, denoted by $N(\mathcal{A})$, is a Poisson random variable:

$$N(\mathcal{A}) \sim \text{Poisson}(\Lambda), \tag{13}$$

where $\Lambda = \lambda|\mathcal{A}|$, λ is the intensity of the PPP, and $|\mathcal{A}|$ denotes the area of set $\mathcal{A}$. We recall that the Poisson distribution is characterized by the following probability mass function

$$\Pr(N(\mathcal{A}) = k) = \sum_{k=0}^{\infty} \frac{\mathrm{e}^{-\Lambda}\Lambda^k}{k!}\delta(N(\mathcal{A}) - k). \tag{14}$$

- From the previous result, the number of APs in the γ-neighborhood of any user adopting the pilot sequence $\mathbf{x}_p^{(p)}$ is a random variable which follows the Poisson distribution with parameter $\Lambda_R = \pi\gamma^2\lambda_R$. We observe that this random number coincides also with the edge degree of the corresponding variable node.

- Similarly, the number of users adopting the pilot sequence $\mathbf{x}_p^{(p)}$ in the γ-neighborhood of any AP follows the Poisson distribution with parameter $\Lambda_T^{(p)} = \pi\gamma^2\lambda_T^{(p)}$. This number coincides also with the edge degree of the corresponding factor node.

- From the property of the Poisson distribution, we observe that asymptotically, a fraction of users in $\mathcal{G}_p$ equal to $\mathrm{e}^{-\Lambda_R}$ has an empty γ-neighborhood and they are not served by the network. We exclude such users from our analysis and consider only users with a non-empty γ-neighborhood.

- As the average number of edges originated in a variable node is $\pi\gamma^2\lambda_R$ and the average number of users in a PPP network of area D^2 is $\lambda_T^{(p)}$, the average number of edges in the corresponding bipartite graphs is $\pi\gamma^2\lambda_R\lambda_T^{(p)}D^2$. Interestingly, the average number of edges scales linearly with the average number of variable and factor nodes $\lambda_T^{(p)}D^2$ and $\lambda_R D^2$, respectively. Thus, the bipartite random graph is *sparse.*

The bipartite random graphs generated as described above, which model CF MaMIMO systems, are characterized by independently generated nodes, while the edges exhibit intrinsic correlations induced by the underlying geometric constraints. They are known in the literature as bipartite geometric graphs [8] or AB geometric graphs [9].

6 Asymptotic Identifiability for PPP Networks

In this section we investigate the identifiability properties of the standard ensemble of PPP networks characterized by the parameters $(\lambda_T^p, \lambda_R, \gamma)$ in the asymptotic regime where the the network dimensions tend to infinity, i.e., $D \to +\infty$. This analysis is of fundamental interest, as it aims to establish whether the bilinear inverse problem associated with the joint estimation of the channel matrix and the transmitted data, represented by the unknown parameters $\mathbf{H}_I$ and $\mathbf{X}_d$, respectively, is identifiable independently of any particular realization of the CF MaMIMO network. Instead, identifiability is assessed solely as a function of macroscopic network parameters, such as the user and AP spatial intensities and the neighborhood radius.

To address this problem, we leverage the observation that a sufficient identifiability condition is satisfied if the Karp-Sipser procedure yields an empty core in the associated bipartite graph representation of the network. This condition has been intensively studied within the framework of the maximum matching problem on random graphs. Accordingly, we build upon well-establish results from this literature to derive identifiability conditions for PPP-based network models in the large-system limit.

In the following, we review the state-of-the-art results relevant to this problem and subsequently introduce an analytical framework that enables tractable performance characterization. Specifically, by relaxing the edge-dependence induced by the underlying geometric constraints,

we approximate the original PPP-based graph ensemble with an equivalent ensemble of random graphs having the same variable and factor node distributions from an edge perspective as the original PPPbased random graph whle having independent edges. Within this framework, we derive a density evolution algorithm that characterizes the identifiability properties over the resulting network ensemble.

The maximum matching problem on sparse random graphs has a long history, and its relevance here stems from the fact that the leaf-removal (Karp–Sipser) procedure isolates/groups the nodes and edges of the graph that impede full identifiability into a residual subgraph called the *core* where all remaining nodes have degree at least two. In ensembles with independent edges and locally tree-like structure, characterized by a macroscopic parameter, this core appears in a specific range of the parameter and undergoes a sharp transition referred to as core-percolation transition. In other words, graphs characterized by the parameter below a critical threshold have an empty core or eventually a core with a negligible number of nodes whereas above the threshold the graph presents a giant core with a positive fraction of nodes as the number of nodes tends to infinity. The appearance of the core controls the asymptotic matching deficit, i.e., the possibility to cover all graph nodes with maximum matching and the performance of iterative "peeling"-type decoders as the ones adopted in LDPC decoding over the binary erasure channel (BEC). For random bipartite graphs with independent edge-generation mechanisms, such as Bernoulli bipartite graphs or "fixed left-degree" models, there exist sharp asymptotic characterizations of (i) the maximum matching size, (ii) the conditions for left-perfect matchings, and (iii) when the pure leaf-removal phase of Karp–Sipser procedure suffices to reach a maximum matching with high probability. In [10], the analysis of maximum matching in random bipartite graphs motivated by cuckoo hashing is provided and thresholds are also established when greedy/leaf-removal steps are sufficient to achieve maximum matching asymptotically. These results are particularly valuable for our purposes because they support a density-evolution-style analysis: the local neighborhood of a uniformly chosen node converges (in distribution) to a branching process, enabling closed-form fixed-point equations for the survival probability of nodes under leaf removal. Recently substantial progress has been achieved in describing the size and fluctuations of Karp–Sipser cores and matching numbers in sparse locally tree-like ensembles such as Erdös–Rényi graphs and configuration models by [11]. Although our graph is geometric and thus not independent-edge, these works provide the reference point for the type of asymptotic performance characterization targeted here.

In contrast, for bipartite random geometric graphs (BRGGs), including those induced by two independent spatial PPPs such as those considered in this work, edges are not independent because they are induced by geometry. This introduces long-range dependencies through the overlap of neighborhoods and spatial clustering, and consequently many of the sharp "locally tree-like" statements available for Erdös-Rényi and configuration ensembles do not transfer directly. Specifically, for core-percolation / greedy leaf removal, most sharp results are currently available for independent-edge or locally tree-like ensembles. There is a significant body of work connecting core structure to algorithmic peeling and controllability in bipartite/directed settings, reinforcing the conceptual link between the emergence of a core and the impossibility of fully eliminating variables by local constraints. However, for geometric bipartite graphs, a comprehensive theory describing the Karp-Sipser core size, its threshold behavior, and the induced matching deficit (as functions of $\lambda_T^p, \lambda_R, \gamma$)) is, at present, substantially less developed, precisely because geometric dependence breaks the standard branching-process decoupling that is essential to classical density evolution.

The above literature suggests a clear methodological direction. On the one hand, the Karp-Sipser/peeling paradigm is the natural bridge to identifiability, since identifiability is guaranteed when leaf-removal eliminates the graph completely (empty core). On the other hand, the geometric bipartite random geometric graph introduces dependencies that obstruct direct application of classical mean-field analyses. Therefore, following a standard strategy in spatial networks, we proceed by relaxing edge dependence and approximating the original model by an analytically tractable independent-edge PPP-based ensemble with matched macroscopic statistics (e.g., mean degrees as functions of $\lambda_T^p, \lambda_R, \gamma$). Within this surrogate ensemble, the local neighborhood becomes tree-like

in the limit, enabling a density-evolution recursion analogous to LDPC decoding on the BEC [4], where the peeling decoder corresponds to iterative elimination of degree-one check nodes and the residual stopping set corresponds to the Karp-Sipser-core.

Thus, we leverage on the analysis of LDPCs based on density evolution, see, e.g., [4]. The identifiability conditions for a PPP network are summarized in the following proposition.

Proposition 3. IDENTIFIABILITY CONDITIONS FOR A PPP NETWORK– *Consider a PPP network with AP spatial intensity λ_R and user spatial intensity $\lambda_T^{(p)}$ for the users in group $\mathcal{G}_p$ transmitting the pilot sequence $\mathbf{x}_p^{(p)}$. Define*

$$\Lambda_T^{(p)} = \lambda_T^{(p)} \pi \gamma^2$$

as the average number of users in $\mathcal{G}_p$ within a γ-neighborhood of a given AP, and

$$\Lambda_R = \lambda_R \pi \gamma^2$$

as the average number of APs within the γ-neighborhood of a given user.

Assume that users and APs with empty γ-neighborhood are neglected and that edges in the associated bipartite graph are generated independently. Then, in the asymptotic regime where the network dimension $D \to +\infty$, the unknown parameters $\mathbf{h}_I$ and $\mathbf{X}_d$ are identifiable if for each group of users $\mathcal{G}_p$, the following fixed point equation

$$z = \epsilon_\Delta \mathrm{e}^{-\Lambda_R w(z)}$$

with

$$w(z) = \mathrm{e}^{-\Lambda_T^{(p)} z}, \tag{15}$$

and

$$\epsilon_\Delta = \frac{1 - (1 + \Lambda_T^{(p)}) \mathrm{e}^{-\Lambda_T^{(p)}}}{1 - \mathrm{e}^{-\Lambda_T^{(p)}}} \tag{16}$$

admits no nonzero solution for z in the interval $(0, \epsilon_\Delta]$.

Proof: From the proof of Proposition 1, the identifiability conditions should be satisfied for each group of users $\mathcal{G}_p$ using the same pilot sequence and thus we focus on the users in $\mathcal{G}_p$.

We associate a variable node and a factor node of a factor graph to each user in $\mathcal{G}_p$ and each AP in the PPP network, respectively. According to the construction of the bipartite graph of the PPP-based network, a variable node has degree k with probability $\mathrm{e}^{-\Lambda_R} \frac{\Lambda_R^k}{k!}$, while a factor node has degree k with probability $\mathrm{e}^{-\Lambda_T^{(p)}} \frac{\Lambda_T^{(p)k}}{k!}$. In the following we consider a surrogate factor graph with the same variable and factor degree distribution from a node perspective but with edges independently randomly generated without enforcing the geometrical constraints of the bipartite geometric graph. Additionally, we remove all the variable and factor nodes with zero degree since they are intrinsically inactive in the network. This implies a new normalization of the variable node and factor node distributions from a node perspective and thus a variable node has degree k with probability $L_R(k) = \frac{\mathrm{e}^{-\Lambda_R}}{1 - \mathrm{e}^{-\Lambda_R}} \frac{\Lambda_R^k}{k!}$, while a factor node has degree k with probability $L_T^{(p)}(k) = \frac{\mathrm{e}^{-\Lambda_T^{(p)}}}{1 - \mathrm{e}^{-\Lambda_T^{(p)}}} \frac{\Lambda_T^{(p)k}}{k!}$. Considering the sparsity of the graph discussed in Section 5, it is straightforward to verify that the message passing algorithm for identifiability verification described in Section 4 over the surrogate graph is equivalent to a message passing algorithm decoding of a LDPC for binary erasure channels and the average erasure probability is is determined via the density evolution function. Then, the problem reduces to determine the variable and factor node degree distributions from an edge perspective, see, e.g., [4] in order to compute the density evolution function. Based on the variable and factor node degree distributions from a node perspective, it

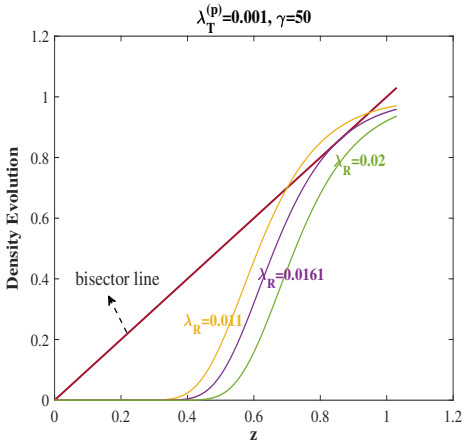

Figure 1: Density evolution function

is possible to show that the variable degree distribution from an edge perspective is given by

$$\xi(x) = \frac{\sum_{i=1}^{+\infty} kL_R(k)x^{k-1}}{\sum_{i=1}^{+\infty} kL_R(k)} = \mathrm{e}^{-\Lambda_R(1-x)}$$

and similarly, the factor node degree distribution from an edge perspective is given by

$$\tau(x) = \mathrm{e}^{-\Lambda_T^{(p)}(1-x)}.$$

The density evolution function can be derived along the lines of Theorem 3.50 in [4] and we obtain that the average probability that a variable node k cannot identify its channel $\mathbf{h}_{I,k}$ at iteration ℓ and forwards Δ to its γ-neighborhood is given by

$$z^{(\ell)} = \epsilon_\Delta \xi(1 - \tau(1 - z^{(\ell-1)})) \tag{17}$$

$$= \epsilon_\Delta \mathrm{e}^{-\Lambda_R w(z^{(\ell-1)})} \tag{18}$$

where $\epsilon_\Delta = (1-(1+\Lambda_T^{(p)})\mathrm{e}^{-\Lambda_T^{(p)}})/(1-\mathrm{e}^{-\Lambda_T^{(p)}})$ is the fraction of nodes that forward Δ in the initial step, and $w(\cdot)$ is defined in (15). The final formulation of Proposition 3 enforces the conditions for the convergence to zero of $z^{(\ell)}$ for ℓ sufficiently large (see, e.g., Theorem 3.59 in [4]), which is equivalent to a Karp-Sipser empty core. ■

In Fig. 1, we plot the density evolution assuming a neighborhood $\gamma = 50$ and a user spatial intensity $\lambda_T^{(p)} = 10^{-3}$ and various values of the AP spatial intensity, namely, $\lambda_R = \{0.011, 0.0161, 0.02\}$. For the parameters considered in this example, $\Lambda_T^{(p)} = 7.854$ and $\epsilon_\Delta = 0.99695$, that is, in the initial iteration of the message passing algorithm only 0.305% of the user channels is identifiable. However, after a sufficient number of the iterations, all the channel can be correctly identified for $\lambda_R = 0.02$, while only a very limited fraction of channels are identifiable for $\lambda_R = 0.01$ and the identifiability condition is not satisfied. Finally, for a AP spatial intensity $\lambda_R = 0.0161$, a limiting condition appears and a very large number of iterations are required to identify all user parameters in the system. In the following section we will discuss the behaviour of the density evolution as a function of z, $\Lambda_T^{(p)}$, and Λ_R in greater details.

Remark 3. *As byproduct, the derivation of Proposition 3 enables to determine the average probability that a user belongs to one of the subsequent subsets $\mathcal{G}_{p,1}$, $\mathcal{G}_{p,2}$, ... or equivalently the fraction of users belonging to each of the subsets and eventually, in the case that the system is not identifiable, the fraction of users whose channel cannot be identified. More specifically, the average probability that a user $k \in \mathcal{G}_p$ belongs to $\mathcal{G}_{p,s}$ is $\mathbb{P}\{k \in \mathcal{G}_{p,s} | k \in \mathcal{G}_p\} = z^{(s-1)} - z^{(s)}$, with $s = 1, 2, \ldots$, $z^{(0)} = 1$,*

$z^{(1)} = \epsilon_\Delta$, *and* $z^{(s)} = \epsilon_\Delta \mathrm{e}^{-\Lambda_R w(z^{(s-1)})}$ *with* $w(z^{(s)}) = \mathrm{e}^{-\Lambda_T^{(p)} z^{(s)}}$. *For a PPP network where the identifiability conditions are not satisfied, the probability that the channel of user* k *is not identifiable is* $\mathbb{P}\{k \notin \mathcal{G}_{p,n}, n = 1, 2, \dots s\} = z^{(s+1)}$ *with* $z^{(s+1)} = z^{(s)}$.

Remark 4. *For a given intensity of the AP PPP* λ_R*, Proposition 3 enables to determine the maximum intensity* $\lambda_T^{(p)}$ *for which the PPP network is still identifiable and thus the minimum number of pilot sequences* P *necessary for the identifiability of a given PPP network with given intensities* λ_R *and* λ_T*. Alternatively, given* λ_R *and the maximum number of pilot sequences* P*, Proposition 3 allows to determine the maximum intensity of the user PPP* λ_T *under the constraint of identifiability.*

These observations suggest that, in the asymptotic regime, it is possible to identify a region in the parameter space (λ_T, λ_R) for which realizations of the PPP-based network ensemble are identifiable with high probability, as well as a complementary region where identifiability is not guaranteed. We refer to the former as the identifiability region of the PPP-based network and examine its properties in greater detail in the following section.

7 On the identifiability region of a PPP network

In this section, we analyze the behavior of the density evolution and highlight the presence of phase-transition phenomena in the identifiability properties of the system parameters. In particular, for each fixed value of Λ_T^p, there exists a critical (transition) value Λ_R^* for the APs such that the system undergoes a sharp transition from a non-identifiable to an identifiable regime. Specifically, for $\Lambda_R > \Lambda_R^*$, the density evolution converges to zero corresponding to an identifiable system, whereas for $\Lambda_R < \Lambda_R^*$, the density evolution converges to a strictly positive fixed point and the system remains non-identifiable.

Similarly, for a fixed value of Λ_R, there exists a critical threshold $\Lambda_T^{(p)*}$ governing a complementary phase transition. In this case, the system is identifiable when $\Lambda_T^{(p)} < \Lambda_T^{(p)*}$, while increasing $\Lambda_T^{(p)}$ beyond this critical value drives the system into a non-identifiable phase.

Together, these critical thresholds determine a well-defined identifiability phase diagram in the $(\Lambda_T^{(p)}, \Lambda_R)$, parameter plane, partitioning the space into identifiable and non-identifiable regions in the large-system limit.

For a given spatial intensity pair $(\lambda_T^{(p)}, \lambda_R)$ and radius γ, we define the function $f(\Lambda_T^{(p)}, \Lambda_R, z)$ as follows

$$f(\Lambda_T^{(p)}, \Lambda_R, z) = \frac{1 - (1 + \Lambda_T^{(p)})\mathrm{e}^{-\Lambda_T^{(p)}}}{1 - \mathrm{e}^{-\Lambda_T^{(p)}}} \mathrm{e}^{-\Lambda_R \mathrm{e}^{-\Lambda_T^{(p)} z}} \tag{19}$$

The fixed point of this recursion characterize the asymptotic behavior of the peeling (Karp-Sipser) procedure and, consequently, the identifiability properties of the system.

We now characterize key monotonicity properties of the function $f(\Lambda_T^{(p)}, \Lambda_R, z)$, which are fundamental to the existence of sharp identifiability threshold.

Lemma 1. *Let* $\Lambda_T^{(p)} > 0$, $\Lambda_R >$, *and* $z \in [0, 1]$. *Consider the function* $f(\Lambda_T^{(p)}, \Lambda_R, z)$ *in (19) Then, the following properties hold:*

- $f(\Lambda_T^{(p)}, \Lambda_R, z)$ *is strictly decreasing in* Λ_R *over* $(0, +\infty)$.
- $f(\Lambda_T^{(p)}, \Lambda_R, z)$ *is strictly decreasing in the interval* $z \in [0, +1]$.
- $f(\Lambda_T^{(p)}, \Lambda_R, z)$ *is increasing in* $\Lambda_T^{(p)}$ *in the interval* $(0, +\infty)$.

Proof: The results follow directly by computing the derivatives of $f(\Lambda_T^{(p)}, \Lambda_R, z)$ with respect to Λ_R, z and $\Lambda_T^{(p)}$, which are respectively negative, positive and non negative, over the specified domains.

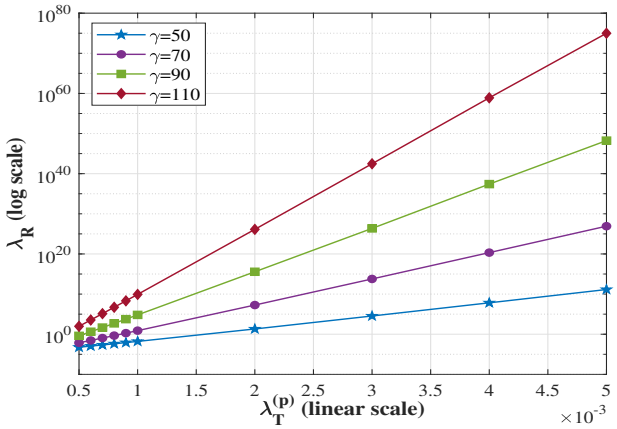

Figure 2: Identifiability region λ_R versus $\lambda_T^{(p)}$ for various radius γ

■

The monotonicity of $f(\Lambda_T^{(p)}, \Lambda_R, z)$ with respect to $\Lambda_T^{(p)}$, for a given value of Λ_R guarantees the existence of a *critical value* ${\Lambda_T^{(p)}}^*$, such that the system is identifiable for $\lambda_T^{(p)} < {\lambda_T^{(p)}}^*$ and is not identifiable for $\lambda_T^{(p)} > {\lambda_T^{(p)}}^*$.

From a practical standpoint, it is of interest to determine the threshold for $\Lambda_T^{(p)*}$ that discriminates identifiable and non-identifiable regimes. To this end, we fix Λ_R and characterize the critical point as the value at which the density evolution function becomes tangent to the bisector line. This condition is captured by the following system of equations

$$f(\Lambda_T^{(p)}, \Lambda_R, z) = z \quad \text{and} \quad \frac{\partial f(\Lambda_T, \Lambda_R, z)}{\partial z} = 1. \tag{20}$$

Equivalently, ${\lambda_T^{(p)}}^*$ corresponds to the point at which $f(\Lambda_T^{(p)}, \Lambda_R, z) - z$ tangentially touches the horizontal axis ($z = 0$) or equivalently $f(\Lambda_T^{(p)}, \Lambda_R, z)$ tangentially touches the bisector line. The system equation solution $({\Lambda_T^{(p)}}^*, z^*)$ must additionally satisfy $z^* \leq \epsilon_\Delta({\Lambda_T^{(p)}}^*)$.

An analogous reasoning applies when we fix $\Lambda_T^{(p)}$. In this case, there exists a critical value Λ_R^* such that the system is not identifiable for $\Lambda_R < \Lambda_R^*$ and identifiable for $\Lambda_R > \Lambda_R^*$. the system is identifiable. The threshold Λ_R^*, is obtained by solving the same system of equations in (20) with respect to Λ_R.

Rather than expressing identifiability conditions solely in terms of the average number of users or APs within a γ-neighborhood, from a system design perspective, it is of practical interest to describe the system in terms of user and AP spatial intensities $\lambda_T^{(p)}$ and λ_R along with the neighborhood radius γ, since they are independent parameters. The system of equations in (20) can therefore be equivalently expressed in terms of $\lambda_T^{(p)}, \lambda_R$ and γ leading to an equivalent characterization of the identifiability region.

To study the impact of the neighborhood radius γ, we determine the identifiability region in the $(\lambda_T^{(p)}, \lambda_R)$ plane for different values of the parameter γ.

The system of equations in (20) is solved numerically by applying the Newton method. For each value of $\lambda_T^{(p)}$, the corresponding critical value λ_R^* is determined for various radii γ. The user intensity $\lambda_T^{(p)}$ is varied over the range $\lambda_T^{(p)} = [0.0005, 0.005]$.

Fig. 2 illustrates the resulting identifiability regions. All the points lying above the curve correspond to identifiable systems while the points below the curve correspond to non-identifiable configurations. As we can observe in Fig. 2, the feasible region is convex and becomes larger as the radius γ decreases. The vertical axis is plotted on logarithmic scale.

8 References

[1] R. Gholami, L. Cottatellucci, and D. Slock, "Tackling pilot contamination in cell-free massive MIMO by joint channel estimation and linear multi-user detection," in *Proc. IEEE Int. Symp. Inf. Theory (ISIT)*, 2021.

[2] C. Forsch and L. Cottatellucci, "Expectation propagation for distributed inference in grant-free cell-free massive MIMO," *submitted to IEEE Trans. Signal Process.*, 2025, preprint available at: https://arxiv.org/abs/2601.04166.

[3] E. de Carvalho and D. T. Slock, "Blind and semi-blind FIR multichannel estimation:(global) identifiability conditions," *IEEE Transactions on Signal Processing*, vol. 52, no. 4, pp. 1053–1064, 2004.

[4] T. Richardson and R. Urbanke, *Modern coding theory*. Cambridge university press, 2008.

[5] R. M. Karp and M. Sipser, "Maximum matching in sparse random graphs," in *22nd Annual Symposium on Foundations of Computer Science (sfcs 1981)*. IEEE, 1981, pp. 364–375.

[6] J. Aronson, A. Frieze, and B. G. Pittel, "Maximum matchings in sparse random graphs: Karp–sipser revisited," *Random Structures & Algorithms*, vol. 12, no. 2, pp. 111–177, 1998.

[7] J.-H. Zhao and H.-J. Zhou, "Two faces of greedy leaf removal procedure on graphs," *Journal of Statistical Mechanics: Theory and Experiment*, vol. 2019, no. 8, p. 083401, 2019.

[8] M. Penrose, *Random geometric graphs*. OUP Oxford, 2003, vol. 5.

[9] M. D. Penrose, "Continuum ab percolation and ab random geometric graphs," *Journal of Applied Probability*, vol. 51, no. A, pp. 333–344, 2014.

[10] A. Frieze and P. Melsted., "Maximum matchings in random bipartite graphs and the space utilization of cuckoo hash tables," *Random Structures Algorithms*, vol. 41, no. 3, pp. 334–364, 2012.

[11] M. Glasgow, M. Kwan, A. Sah, and M. Sawhney, "A central limit theorem for the matching number of a sparse random graph," *arXiv preprint arXiv:2402.05851*, 2024.